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Short Communication

# Coupled vibration of isotropic metal hollow cylinders with large geometrical dimensions

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**Abstract**

In this paper, the coupled vibration of isotropic metal hollow cylinders with large geometrical dimensions is studied by using an approximate analytic method. According to this method, when the equivalent mechanical coupling coefficient that is defined as the stress ratio is introduced, the coupled vibration of a metal hollow cylinder is reduced to two equivalent one-dimensional vibrations, one is an equivalent longitudinal extensional vibration in the height direction of the cylinder, and the other is an equivalent plane radial vibration in the radius direction. These two equivalent vibrations are coupled to each other by the equivalent mechanical coupling coefficient. The resonance frequency equation of metal hollow cylinders in coupled vibration is derived and longitudinal and radial resonance frequencies are computed. For comparison, the resonance frequencies of the hollow cylinders are also computed by using numerical method. The analysis shows that the results from these two methods are in a good agreement with each other.


## 1. Introduction

In underwater acoustics and high power ultrasonics, sandwich piezoelectric ultrasonic composite transducers, which are also known as Langevin piezoelectric composite ultrasonic transducers, are widely used as large power sound radiators. This kind of composite ultrasonic transducers usually consist of a number of thickness-polarized piezoelectric ceramic rings which are sandwiched between two metal masses, the two front and back metal masses may be solid or hollow metal cylinder. According to traditional one-dimensional (1D) design theory of this kind of transducers [1–5], it is required that the lateral or radial dimensions of the transducer must be far less than its longitudinal dimension. Generally speaking, when the radial dimensions are less than a quarter of the longitudinal wavelength, 1D theory can be used and the measured resonance frequencies of the transducer are in a good agreement with the theoretical results.

However, along with the development of ultrasonic technology, ultrasonic transducers are being used in more and more new applications, such as high frequency ultrasonic metal and plastic welding and some practical applications concerning very large ultrasonic power. In these cases, the radial dimensions of the transducer are usually larger than a quarter of the longitudinal wavelength. Therefore, traditional 1D

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longitudinal design theory is no longer applicable for sandwich transducers with large radial dimensions. Specifically speaking, the following three cases should be given special attention in their theoretical design and calculation. (1) High frequency sandwich piezoelectric composite transducers such as those used in ultrasonic metal welding. When the resonance frequency of the sandwich transducer is increased, the longitudinal wavelength and geometrical dimension will decrease accordingly. According to the assumptions introduced in 1D longitudinal theory, the radial dimensions of the transducer must also be decreased. Therefore, the cross section of the transducer is small and the mechanical strength and the power capacity will be lowered. To raise the mechanical strength and the power capacity, the radial dimensions of the high frequency sandwich transducer must be increased; thus, the radial vibration in this kind of transducers must be considered and the coupled vibration of the transducer should be analyzed. (2) High power sandwich composite transducers. In some ultrasonic applications, such as ultrasonic metal forming and ultrasonic plastic welding, very large ultrasonic power is needed; therefore, the radial dimensions exceed a quarter of the longitudinal wavelength and 1D theory is also not applicable. (3) In recent years, a new type of high power ultrasonic radiators are used in ultrasonic cleaning and ultrasonic sonochemistry, the ultrasonic radiator is a metal tube with large radial dimension. When it is excited by a sandwich piezoelectric transducer longitudinally, the radiator will vibrate in both longitudinal and radial direction because of the Poisson's effect; therefore, large power can be given out. In the above-mentioned three cases, the vibration of the transducer or the radiator is a coupled one of longitudinal and radial vibrations. Therefore, new design theory must be developed in order to study the coupled vibration of the sandwich transducers with large cross section or high resonance frequency.

For the coupled vibration of isotropic metal rod and hollow cylinder, a number of research works can be found in previous literatures. Rayleigh and Love studied the sound velocity of longitudinal vibration in rods and obtained the corrected sound velocity and the dispersion equation [6,7]. Mindlin analyzed the axisymmetric vibration of disks using complex second-order approximate theory and obtained the dispersion equation [8,9]. At the same time, numerical methods are also used in the vibration analysis of elastic cylinders and disks [10,11].

An approximate analytical method known as apparent elasticity method or equivalent elastic method has also been used to analyze the coupled vibration of ultrasonic vibrating system [12–14]. In this paper, the hollow cylinder ultrasonic radiator with large lateral and longitudinal dimensions is analyzed by using the equivalent elastic method. The resonance frequency equations are derived analytically and numerical methods are also used to analyze the coupled vibration. It is illustrated that the results from the analytical method are in good agreement with those from numerical method.

## 2. coupled vibration of an isotropic metal hollow cylinder with large lateral and longitudinal dimensions

Fig. 1 illustrates a large-dimension metal hollow cylinder. This means that the cylinder has a large lateral geometrical dimension compared with its longitudinal dimension; i.e., the radius is comparable to its length. In this case, the vibration of the cylinder is more complex than that of a slender cylinder or a thin ring. For a slender cylinder whose length is much larger than its radius, its vibration is a simple 1D longitudinal vibration.

In the figure, $r$ and $Z$ are radial and axial coordinates. The height, the inner and the outer radius of the cylinder are $h$, $b$ and $a$, respectively, and $a$ is comparable to $h$. In the analysis, the cylindrical coordinate is used and the direction of the height of the cylinder is along that of the $Z$-axis. When the height of the cylinder is much larger or less than its radius, the vibration of the cylinder can be reduced to 1D longitudinal vibration of a slender cylinder or plane radial vibration of a thin circular ring. For these cases, the problem is simple and 1D theory can be used. However, when the dimensions of the cylinder do not meet the requirement of 1D theory, the equations describing the coupled vibration of finite-dimension cylinders must be solved. Because of the complexity of the coupled vibration equations, the analytical solutions are difficult to obtain. In the following analysis, the equivalent mechanical coupling coefficient is introduced and the coupled vibration of large-dimension hollow cylinders is studied by using an approximate analytic method when the shearing stresses and strains are ignored.

The basic principle of the approximate analytic method is outlined as follows. When the metal hollow cylinder radiator is excited to vibrate in the longitudinal direction, the radial extensional vibration is produced because of Poisson's effect; the vibration of the radiator is a complex 3D coupled vibration, and its analytical

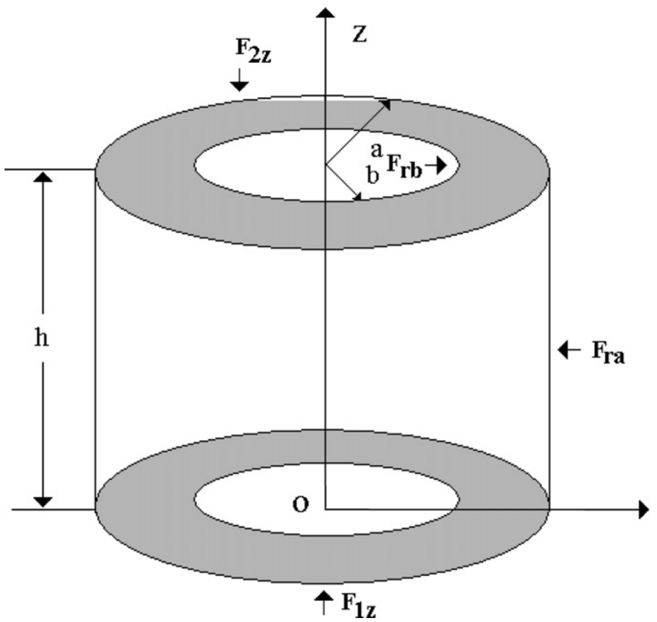


Fig. 1. A metal hollow cylinder ultrasonic radiator with large dimension.

solutions are difficult to obtain. To simplify the analysis, the shearing stress and strain in the cylinder are ignored, and it is assumed that only extensional vibrations are considered. When equivalent elastic constants and equivalent mechanical coupling coefficient are introduced, the complex axisymmetrical coupled vibration of the metal hollow cylinder radiator with large dimension can be reduced to two equivalent extensional vibrations, one is the longitudinal extensional vibration in the $Z$-axis direction, and the other is the plane radial vibrations in the radial direction. However, these two equivalent extensional vibrations are not independent of each other; they are coupled together by means of the equivalent mechanical coupling coefficients that are defined as the ratio of the mechanical stresses in different axial directions. It should be noted that though the two introduced equivalent extensional vibrations in the cylinder are similar to traditional 1D vibrations, they are entirely different. These two equivalent extensional vibrations have different equivalent elastic constants. These equivalent elastic constants not only depend on the material parameters, but also on the geometrical dimensions and the coupling of the vibrations in different directions.

Based on the theory of elastic dynamics, for the axisymmetrical coupled vibration of a metal hollow cylinder, when the shearing stresses and strains are ignored, we have, $\xi_\theta = 0$, $T_{r\theta} = T_{rz} = T_{\theta z} = 0$, $S_{r\theta} = S_{\theta z} = S_{rz} = 0$. In this case, the following equations can be obtained:

$$\rho \frac{\partial^2 \xi_r}{\partial t^2} = \frac{\partial T_r}{\partial r} + \frac{T_r - T_\theta}{r}, \tag{1}$$

$$\rho \frac{\partial^2 \xi_z}{\partial t^2} = \frac{\partial T_z}{\partial z}. \tag{2}$$

Here, $\rho$ is the volume density of the metal hollow cylinder, $\xi_r$, $\xi_\theta$, $\xi_z$ are three displacement components in the radial, tangential and height directions, $T_r$, $T_\theta$, $T_z$, $T_{r\theta}$, $T_{rz}$, $T_{\theta z}$ and $S_r$, $S_\theta$, $S_z$, $S_{r\theta}$, $S_{rz}$, $S_{\theta z}$ are stress and strain components in the metal hollow cylinder. In cylindrical coordinates, the relationship between the strain and the displacement are as follows:

$$S_r = \frac{\partial \xi_r}{\partial r}, \quad S_\theta = \frac{\xi_r}{r}, \quad S_z = \frac{\partial \xi_z}{\partial z}, \tag{3}$$

Based on Hooker's law, the relationship between the strain and the stress are expressed as

$$S_r = \frac{1}{E}[T_r - v(T_\theta + T_z)], \tag{4}$$

$$S_\theta = \frac{1}{E}[T_\theta - v(T_r + T_z)], \tag{5}$$

$$S_z = \frac{1}{E}[T_z - v(T_\theta + T_r)]. \tag{6}$$

Here, $E$ and $v$ are Young's modulus and Poisson's ratio. Let $n = T_z/(T_r + T_\theta)$, which is defined as the equivalent mechanical coupling coefficient. Eqs. (4)–(6) can be rewritten as the following forms:

$$S_r - S_\theta = \frac{1+v}{E}(T_r - T_\theta), \tag{7}$$

$$S_r + S_\theta = \frac{1 - v - 2vn}{E}(T_r + T_\theta), \tag{8}$$

$$S_z = \frac{1 - v/n}{E} T_z. \tag{9}$$

After the above transformations, it can be seen that when the equivalent mechanical coupling coefficient is introduced, the coupled vibration of the metal hollow cylinder can be reduced to two equivalent 1D vibrations, one is the equivalent longitudinal vibration which is described by Eqs. (2) and (9), the other is the equivalent planar radial vibration which is described by Eqs. (1), (7) and (8). However, it should be noted that these two equivalent vibrations are not independent; they are coupled to each other by the equivalent mechanical coupling coefficient. In the following analysis, these two equivalent vibrations will be analyzed, respectively.

### 2.1. Equivalent plane radial vibration of a metal hollow cylinder in coupled vibration

From Eqs. (7) and (8) we have

$$T_r - T_\theta = \frac{E}{1+v}(S_r - S_\theta), \tag{10}$$

$$T_r = \frac{E}{2}\left[\frac{S_r - S_\theta}{1+v} + \frac{S_r + S_\theta}{1 - v - 2vn}\right]. \tag{11}$$

Substituting the expressions of $T_r$ and $T_r - T_\theta$ into Eq. (1) and using Eq. (3) yields

$$\frac{\partial^2 \xi_r}{\partial t^2} = V_r^2\left(\frac{\partial^2 \xi_r}{\partial r^2} + \frac{1}{r}\frac{\partial \xi_r}{\partial r} - \frac{\xi_r}{r^2}\right). \tag{12}$$

Here, $V_r^2 = E_r/\rho$, $E_r = E(1-vn)/(1+v)(1-v-2vn)$. $V_r$ is defined as the equivalent radial speed of sound, and $E_r$ is equivalent radial elastic constant. For harmonic vibration, substituting the radial displacement component $\xi_r = \xi_{r0}\exp(\mathrm{j}\omega t)$ into Eq. (12) yields

$$\frac{\mathrm{d}^2 \xi_{r0}}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}\xi_{r0}}{\mathrm{d}r} - \frac{\xi_{r0}}{r^2} + k_r^2 \xi_{r0} = 0. \tag{13}$$

Here $\xi_{r0}$ is a function of radial coordinate, $k_r = \omega/V_r$, $k_r$ is called the equivalent radial wavenumber. It is obvious that Eq. (13) is Bessel equation of order one, its solution is

$$\xi_{r0} = [AJ_1(k_r r) + BY_1(k_r r)]. \tag{14}$$

Here $A$ and $B$ are constants, $J_1(k_r r)$ and $Y_1(k_1 r)$ are Bessel functions. When the metal hollow cylinder is free from external radial forces, the boundary conditions for the cylinder is $F_{ra} = -F_r|_{r=a} = 0$,

$F_{rb} = -F_r|_{r=b} = 0$. Here $F_r = T_r \cdot S_s$ is radial force, $S_s = 2\pi rh$ is the side area of the cylinder. Substituting Eq. (14) into Eqs. (3) and (11) yields

$$T_r = \frac{E}{2}\left\{\frac{A[k_r J_0(k_r r) - 2J_1(k_r r)/r] + B[k_r Y_0(k_r r) - 2Y_1(k_r r)/r]}{1+v} + \frac{Ak_r J_0(k_r r) + Bk_r Y_0(k_r r)}{1-v-2vn}\right\}. \tag{15}$$

Using the boundary conditions of the metal hollow cylinder in the radial direction, the following expressions can be obtained:

$$AJ(a) + BY(a) = 0, \tag{16}$$

$$AJ(b) + BY(b) = 0, \tag{17}$$

where $J(a) = J(x)|_{x=a}$, $J(b) = J(x)|_{x=b}$, $Y(a) = Y(x)|_{x=a}$, $Y(b) = Y(x)|_{x=b}$. $J(x)$ and $Y(x)$ are two introduced functions, their expressions are

$$J(x) = \left[k_r J_0(k_r x) - 2J_1(k_r x)/x\right](1-v-2vn)/(1+v) + k_r J_0(k_r x), \tag{18}$$

$$Y(x) = [k_r Y_0(k_r x) - 2Y_1(k_r x)/x](1-v-2vn)/(1+v) + k_r Y_0(k_r x). \tag{19}$$

Using Eqs. (16) and (17), the equivalent resonance frequency equation for the equivalent radial vibration of the hollow metal cylinder in coupled vibration can be obtained as

$$J(a)Y(b) - J(b)Y(a) = 0. \tag{20}$$

Eq. (20) can be rewritten as the following form:

$$\frac{k_r a J_0(k_r a) - (1-v-2vn)/(1-vn)J_1(k_r a)}{k_r b J_0(k_r b) - (1-v-2vn)/(1-vn)J_1(k_r b)} = \frac{k_r a Y_0(k_r a) - (1-v-2vn)/(1-vn)Y_1(k_r a)}{k_r b Y_0(k_r b) - (1-v-2vn)/(1-vn)Y_1(k_r b)}. \tag{21}$$

It can be seen that when the coupling between the radial vibration and the longitudinal vibration in the metal hollow cylinder is considered, the equivalent radial resonance frequency equation is different from that of a thin metal ring in planar radial vibration. However, when the equivalent mechanical coupling coefficient becomes zero, Eq. (20) can be reduced to that of a thin metal ring.

On the other hand, it is obvious that the solution to Eq. (21) depends not only on the material parameter and the geometrical dimensions, but also on the equivalent mechanical coupling coefficient. Therefore, the resonance frequency cannot be found only from Eq. (21). Another equation is needed, which is the resonance frequency equation for the equivalent longitudinal vibration in the hollow cylinder and it will be analyzed in the following section of this paper.

### 2.2. Equivalent longitudinal vibration of a metal hollow cylinder in coupled vibration

From Eq. (9) we have

$$T_z = E_z S_z. \tag{22}$$

Here $E_z = E/(1-v/n)$, $E_z$ is called the equivalent longitudinal elastic constant. Substituting Eq. (22) into the longitudinal motion Eq. (2) yields

$$\partial^2 \xi_z/\partial t^2 = V_z^2(\partial^2 \xi_z/\partial z^2). \tag{23}$$

Here, $V_z = (E_z/\rho)^{1/2}$, $V_z$ is defined as the equivalent longitudinal sound of speed. For harmonic motion, $\xi_z = \xi_{z0}\exp(j\omega t)$, Eq. (23) can be reduced to

$$d^2\xi_{z0}/dz^2 + k_z^2 \xi_{z0} = 0. \tag{24}$$

Here $k_z = \omega/V_z$, $k_z$ is called the equivalent longitudinal wavenumber of the equivalent longitudinal vibration of the metal hollow cylinder. The solution of Eq. (24) is

$$\xi_{z0} = A_z \sin(k_z z) + B_z \cos(k_z z). \tag{25}$$

Here $A_z$ and $B_z$ are constants that can be determined by the longitudinal boundary conditions. When the two end surfaces of the metal hollow cylinder are free, the boundary conditions are

$$F_{1z} = -F_z|_{z=0} = 0, \quad F_{2z} = -F_z|_{z=h} = 0. \tag{26}$$

Here, $F_z = T_z S$, $S = \pi(a^2 - b^2)$ is the cross-sectional area of the metal hollow cylinder. Using Eqs. (26) and (3), the resonance frequency equation for the equivalent longitudinal vibration in the metal hollow cylinder can be obtained as

$$\sin(k_z h) = 0. \tag{27}$$

It can be seen that the resonance frequency equation (27) is similar to that of a slender metal hollow cylinder in longitudinal vibration. However, as the equivalent longitudinal wavenumber $k_z$ depends on the equivalent mechanical coupling coefficient, it is impossible to obtain the longitudinal resonance frequency only from Eq. (27).

Eqs. (21) and (27) are the combined resonance frequency equations for a metal cylinder in coupled vibration; they describe the relationship among the material parameter, the geometrical dimensions, the vibrational mode, and the resonance frequencies of the hollow metal cylinder in coupled vibration. By solving Eqs. (21) and (27), it can be seen that two groups of solutions can be found, which are noted as $f_r$, $n_r$ and $f_z$, $n_z$. Considering the practical vibration modes of the cylinder, it can be concluded that these two groups of solutions correspond to two different vibrational modes of the metal cylinder, one is the longitudinal vibration, and the other is the radial vibration. The two resonance frequencies $f_r$ and $f_z$ are the longitudinal and radial resonance frequencies. It should be noted that the frequencies $f_r$ and $f_z$ from Eqs. (21) and (27) are different from those from 1D theory.

From the above analysis, it can be seen that there are two kinds of resonance frequencies for the coupled vibration of the metal hollow cylinder. This is different from the results of 1D theory. According to 1D theory, for a fixed vibrational order, only one frequency can be obtained for a slender cylinder or a thin ring. On the other hand, it can be seen from the solutions of Eqs. (21) and (27) that when the geometrical dimensions satisfy certain conditions, for example, $h \gg a$ or $h \ll a$, the two frequencies from Eqs. (21) and (27) are far away from each other. Therefore, the longitudinal vibration is weakly coupled with the radial vibration. In this case, the vibration of the metal hollow cylinder can be regarded as 1D longitudinal vibration, or plane radial vibration.

Fig. 2 illustrates the theoretical relationship between the equivalent mechanical coupling coefficient and the ratio of length over outer radius of a hollow cylinder radiator when its outer and inner radiuses are fixed. The positive equivalent mechanical coupling coefficient corresponds to the equivalent longitudinal vibration, while

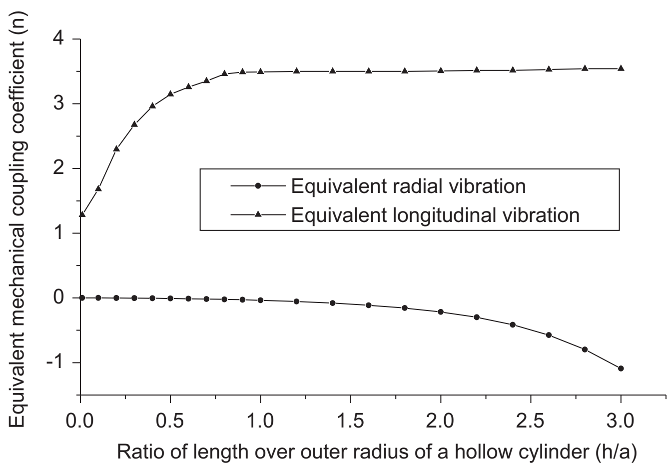


Fig. 2. Theoretical relationship between the equivalent mechanical coupling coefficient and the ratio of length over outer radius of a metal hollow cylinder.

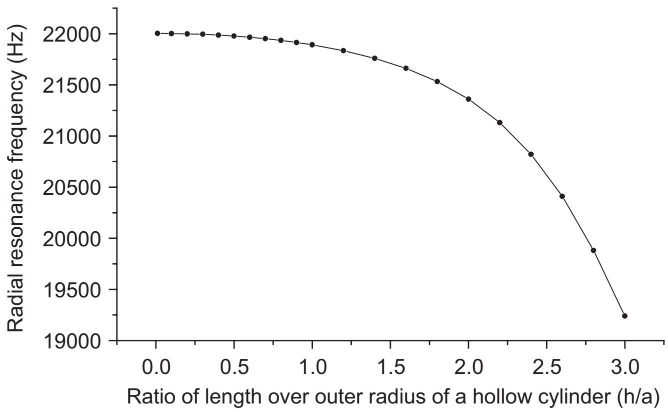


Fig. 3. Theoretical relationship between the radial resonance frequency and the ratio of length over outer radius of a metal hollow cylinder.

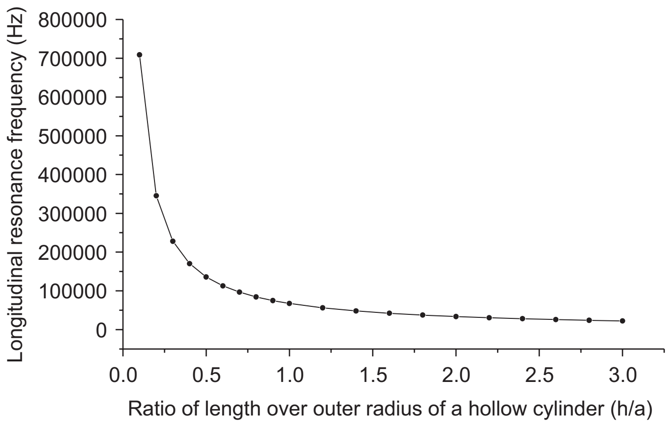


Fig. 4. Theoretical relationship between the longitudinal resonance frequency and the ratio of length over outer radius of a metal hollow cylinder.

the negative equivalent mechanical coupling coefficient corresponds to the equivalent radial vibration. It can be seen that when the length is very small, the negative equivalent mechanical coupling coefficient is very small and remains unchanged; this corresponds to the plane radial vibration of a thin metal ring. When the length is much larger than the outer radius, the positive equivalent mechanical coupling coefficient tends to a constant. This corresponds to the longitudinal vibration of a slender metal cylinder.

Figs. 3 and 4 are theoretical relationship between the radial and longitudinal resonance frequency and the ratio of length over outer radius of a hollow cylinder radiator when the outer and the inner radius are fixed. It can be seen that when the inner and the outer radius are fixed, the radial resonance frequency is decreased when its thickness is increased. The reason is that when the thickness is increased, the longitudinal vibration is produced due to Poisson's effect; the equivalent mass is accordingly increased. For the longitudinal vibration, when the length of the cylinder is increased, the resonance frequency is decreased. The reason is that the longitudinal resonance frequency is inversely proportional to its length.

## 3. Theoretical simulation of the resonance frequency of a metal hollow cylinder with large lateral and longitudinal dimensions

Using the above-developed theory, the resonance frequency of metal hollow cylinders with large lateral and longitudinal dimensions are simulated theoretically. The metal hollow cylinder material used here is steel, its

Table 1
Theoretical resonance frequencies for the metal hollow cylinders in coupled vibration

| $h$ (m) | $a$ (m) | $b$ (m) | $i$ | $j$ | $f_{1r}$ (Hz) | $f_{1z}$ (Hz) | $f_r$ (Hz) | $f_z$ (Hz) | $F_{rn}$ (Hz) | $f_{zn}$ (Hz) |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.12 | 0.0445 | 0.0385 | 1 | 1 | 19888 | 21567 | 18193 | 24570 | 18303 | 24667 |
| 0.12 | 0.054 | 0.048 | 1 | 1 | 16174 | 21567 | 15528 | 23407 | 15567 | 23440 |
| 0.12 | 0.056 | 0.0525 | 1 | 1 | 15192 | 21567 | 14699 | 23221 | 14756 | 23248 |
| 0.12 | 0.057 | 0.051 | 1 | 1 | 15273 | 21567 | 14768 | 23240 | 14902 | 23348 |

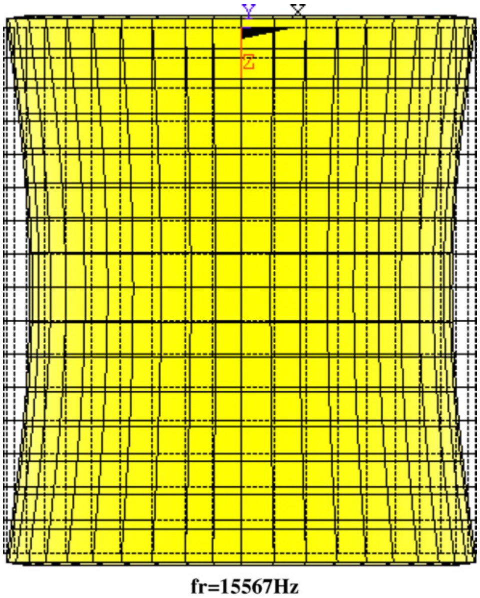


Fig. 5. FEM analysis mode shape for the equivalent radial vibration of a cylinder in coupled vibration.

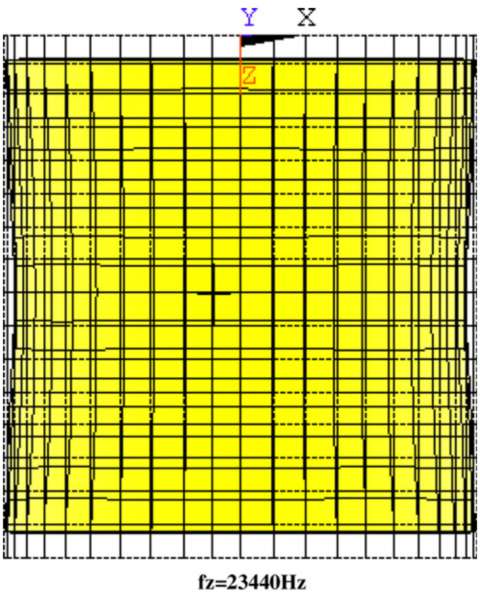


Fig. 6. FEM analysis mode shape for the equivalent longitudinal vibration of a cylinder in coupled vibration.

material parameters are as follows. $\rho = 7800\,\mathrm{kg/m^3}$, $E = 2.09 \times 10^{11}\,\mathrm{N/m^2}$, $\nu = 0.28$. The resonance frequencies of the metal hollow cylinders in coupled vibration are computed by solving the resonance frequency equations. The results are shown in Table 1, where $f_r$ and $f_z$ are theoretical radial and longitudinal fundamental resonance frequencies of the cylinders computed from Eqs. (21) and (27). For comparison, numerical method (here, ANSYS software is used) is used to compute the resonance frequency, and the results are also listed in Table 1. In the table, $f_{rn}$ and $f_{zn}$ are radial and longitudinal resonance frequencies from the numerical method. The analysis mode shapes for the equivalent radial vibration and the equivalent longitudinal vibration of a cylinder with large lateral geometrical dimension in coupled vibration are shown in Figs. 5 and 6. The type of elements is structural-solid-brick 8 nodes 45. It can be seen that the computed frequencies from these two methods are in a good agreement with each other. On the other hand, the radial and longitudinal resonance frequencies $f_{1\mathrm{r}}$ and $f_{1\mathrm{z}}$ from 1D theory are also given in Table 1 for comparison. In the table, $i$ and $j$ correspond to longitudinal and radial vibrational orders of the metal hollow cylinder in coupled vibration.

## 4. Conclusions

The coupled vibration of metal hollow cylinders is analyzed by using an approximate analytic method. The resonance frequency equations are derived and the resonance frequencies are calculated and simulated. To sum up the above analysis, the following conclusions can be drawn.

1. There are two kinds of resonance frequencies for a metal hollow cylinder in coupled vibration, one corresponds to the longitudinal frequency, the other corresponds to the radial frequency, and they are different from the results from 1D theory.
2. The resonance frequencies of some metal hollow cylinders in coupled vibration are obtained by using both analytical and numerical methods. It is shown that the results from the approximate analytic method and numerical methods are in a good agreement with each other.
3. The metal hollow cylinder with large dimensions can be used as a high power ultrasonic radiator.

It can be used in ultrasonic cleaning, ultrasonic liquid processing and sonochemistry. It has the advantage of large radiation area and all-direction ultrasonic radiation.

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